Lecture 26. Linear recurrences and differential equations

Def A linear recurrence is an equation of the form

$$c_0 a_n + c_1 a_{n+1} + \cdots + c_\ell a_{n+\ell} = 0$$

for coefficients $c_0, c_1, \cdots, c_\ell$ and a sequence of numbers $a_0, a_1, a_2, \cdots$.

Note If we take vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \cdots$ given by

$$\vec{x}_n = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+\ell-1} \end{bmatrix},$$

we get a linear dynamical system $\vec{x}_{n+1} = A\vec{x}_n$ with

$$p_A(\lambda) = c_0 + c_1\lambda + \cdots + c_\ell \lambda^\ell$$

e.g. $3a_n - 2a_{n+1} + a_{n+2} = 0$

$$\Longrightarrow a_{n+2} = -3a_n + 2a_{n+1}$$

$$\Longrightarrow \underbrace{\begin{bmatrix} a_{n+1} \\ a_{n+2} \end{bmatrix}}_{\vec{x}_{n+1}} = \begin{bmatrix} a_{n+1} \\ -3a_n + 2a_{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix}}_{\vec{x}_n}$$

$*$ $p_A(\lambda) = \lambda^2 - (0+2)\lambda + (0\cdot 2 - 1\cdot(-3)) = \lambda^2 - 2\lambda + 3$ $(= 3 - 2\lambda + \lambda^2)$

Prop Given a linear recurrence

$$c_0 a_n + c_1 a_{n+1} + \cdots + c_\ell a_{n+\ell} = 0$$

such that the characteristic polynomial $p(\lambda) = c_0 + c_1\lambda + \cdots + c_\ell\lambda^\ell$ has ℓ distinct roots $\alpha_1, \alpha_2, \cdots, \alpha_\ell$, we have

$$a_n = d_1\alpha_1^n + d_2\alpha_2^n + \cdots + d_\ell\alpha_\ell^n$$

for some constants $d_1, d_2, \cdots, d_\ell$.

Def A linear differential equation is an equation of the form

$$c_0 f(t) + c_1 f'(t) + \cdots + c_\ell f^{(\ell)}(t) = 0$$

ℓ^{th} derivative

for coefficients $c_0, c_1, \cdots, c_\ell$ and a differentiable function $f(t)$.

Note If we take a vector function given by

$$\vec{x}(t) = \begin{bmatrix} f(t) \\ f'(t) \\ \vdots \\ f^{(\ell)}(t) \end{bmatrix},$$

we get a matrix differential equation $\vec{x}'(t) = A\vec{x}(t)$ with

$$p_A(\lambda) = c_0 + c_1\lambda + \cdots + c_\ell \lambda^\ell$$

e.g. $5f(t) - 7f'(t) + f''(t) = 0$

$$\Rightarrow f''(t) = -5f(t) + 7f'(t)$$

$$\Rightarrow \underbrace{\begin{bmatrix} f'(t) \\ f''(t) \end{bmatrix}}_{\vec{x}'(t)} = \begin{bmatrix} f'(t) \\ -5f(t) + 7f'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -5 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} f(t) \\ f'(t) \end{bmatrix}}_{\vec{x}(t)}$$

$*$ $p_A(\lambda) = \lambda^2 - (0+7)\lambda + (0\cdot 7 - 1\cdot(-5)) = \lambda^2 - 7\lambda + 5$ $(= 5 - 7\lambda + \lambda^2)$

Prop Given a linear differential equation

$$c_0 f(t) + c_1 f'(t) + \cdots + c_\ell f^{(\ell)}(t) = 0$$

such that the characteristic polynomial $p(\lambda) = c_0 + c_1\lambda + \cdots + c_\ell\lambda^\ell$ has ℓ distinct roots $\alpha_1, \alpha_2, \cdots, \alpha_\ell$, we have

$$f(t) = d_1 e^{\alpha_1 t} + d_2 e^{\alpha_2 t} + \cdots + d_\ell e^{\alpha_\ell t}$$

for some constants $d_1, d_2, \cdots, d_\ell$.

Ex Find the solution of each equation.

(1) The linear recurrence $a_{n+2}=4a_{n+1}-3a_n$ with $a_0=1, a_1=5$.

Sol We may write $3a_n-4a_{n+1}+a_{n+2}=0$

The characteristic polynomial is $p(\lambda)=3-4\lambda+\lambda^2=(1-\lambda)(3-\lambda)$,

which is of degree 2 with 2 distinct roots $\lambda=1,3$.

$\Rightarrow a_n=d_1\cdot 1^n+d_2\cdot 3^n=d_1+d_2\cdot 3^n$ for some constants d_1, d_2

The values $a_0=1$ and $a_1=5$ yield the linear system

$$\begin{cases} d_1+d_2=1 \\ d_1+3d_2=5 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$\Rightarrow d_1=-1, d_2=2 \Rightarrow a_n=\boxed{-1+2\cdot 3^n}$

(2) The differential equation $3f(t)-4f'(t)+f''(t)=0$ with $f(0)=1, f'(0)=5$.

Sol The characteristic polynomial is $p(\lambda)=3-4\lambda+\lambda^2=(1-\lambda)(3-\lambda)$,

which is of degree 2 with 2 distinct roots $\lambda=1,3$.

$\Rightarrow f(t)=d_1e^{1\cdot t}+d_2e^{3t}=d_1e^t+d_2e^{3t}$ for some constants d_1, d_2

$\Rightarrow f'(t)=d_1e^t+3d_2e^{3t}$

The values $f(0)=1$ and $f'(0)=5$ yield the linear system

$$\begin{cases} d_1+d_2=1 \\ d_1+3d_2=5 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & 5 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$\Rightarrow d_1=-1, d_2=2 \Rightarrow f(t)=\boxed{-e^t+2e^{3t}}$

Note The two equations are related via Taylor series

$$f(t) = a_0 + \frac{a_1}{1!}t + \frac{a_2}{2!}t^2 + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{n!}t^n.$$

By taking derivatives, we find

$$\begin{cases} f'(t) = \frac{a_1}{1!} + \frac{a_2}{2!}\cdot 2t + \frac{a_3}{3!}\cdot 3t^2 + \cdots = a_1 + \frac{a_2}{1!}t + \frac{a_3}{2!}t^2 + \cdots = \sum_{n=0}^{\infty} \frac{a_{n+1}}{n!}t^n \\ f''(t) = \frac{a_2}{1!} + \frac{a_3}{2!}\cdot 2t + \frac{a_4}{3!}\cdot 3t^2 + \cdots = a_2 + \frac{a_3}{1!}t + \frac{a_4}{2!}t^2 + \cdots = \sum_{n=0}^{\infty} \frac{a_{n+2}}{n!}t^n \end{cases}$$

$$\Longrightarrow 3f(t) - 4f'(t) + f''(t) = 3\sum_{n=0}^{\infty} \frac{a_n}{n!}t^n - 4\sum_{n=0}^{\infty} \frac{a_{n+1}}{n!}t^n + \sum_{n=0}^{\infty} \frac{a_{n+2}}{n!}t^n$$

$$= \sum_{n=0}^{\infty} \frac{3a_n - 4a_{n+1} + a_{n+2}}{n!}t^n$$

Now the differential equation $3f(t) - 4f'(t) + f''(t) = 0$ converts to the linear recurrence $3a_n - 4a_{n+1} + a_{n+2} = 0$.

Moreover, from the Taylor series we find $f(0) = a_0$ and $f'(0) = a_1$.

Hence the two equations considered in this example are equivalent.

If we substitute the solution of the linear recurrence into the Taylor series, we obtain the expression

$$f(t) = \sum_{n=0}^{\infty} \frac{a_n}{n!}t^n = \sum_{n=0}^{\infty} \frac{-1 + 2\cdot 3^n}{n!}t^n = -\sum_{n=0}^{\infty} \frac{t^n}{n!} + 2\sum_{n=0}^{\infty} \frac{(3t)^n}{n!} = -e^t + 2e^{3t}$$

which recovers the solution of the differential equation.

* This discussion is not essential for Math 313, but is enlightening for understanding how linear algebra can be used to study nonlinear functions.